

MERIDIONAL SURFACES AND $(1, 1)$ -KNOTS

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ABSTRACT. We determine all $(1, 1)$ -knots which admit an essential meridional surface, namely, we give a construction which produces $(1, 1)$ -knots having essential meridional surfaces, and show that if a $(1, 1)$ -knot admits an essential meridional surface then it comes from the given construction.

1. INTRODUCTION

We are interested in finding closed incompressible surfaces or meridional incompressible surfaces in the complement of knots in S^3 . This problem has been studied for many classes of knots, for example for 2-bridge knots [GL],[HT], Montesinos knots [O], alternating knots [M], closed 3-braids [LP],[F], etc.

Let k be a knot in a 3-manifold M . Recall that a surface properly embedded in the exterior of k in M is meridional if its boundary consists of a collection of meridians of k (and possibly some curves on ∂M). A surface S properly embedded in the exterior of k (either closed or meridional), is meridionally compressible if there is a disk D embedded in M , so that $D \cap S = \partial D$ is an essential curve in S (non-contractible in S and non-parallel in S to a meridian of k), such that k intersects D transversely in exactly one point (or in other words, there is an annulus A embedded in the exterior of k , with $\partial A = \partial_0 A \cup \partial_1 A$, such that $A \cap S = \partial_0 A$ and $\partial_1 A$ is a meridian of k); if there is no such a disk, we say that the surface is meridionally incompressible. We say that a meridional surface S is essential if it is incompressible and meridionally incompressible.

A class of knots that has been widely studied is that of tunnel number one knots (a knot k has tunnel number one if there is an arc τ embedded in S^3 , with $k \cap \tau = \partial \tau$ such that $S^3 - \text{int } \eta(k \cup \tau)$ is a genus 2 handlebody). By work of Gordon and Reid [GR], these knots do not admit any planar meridional incompressible surface. However, some of these knots do admit closed incompressible surfaces or meridional incompressible surfaces of genus one or greater. Morimoto and Sakuma [MS] determined all tunnel number one knots whose complements contain an incompressible torus. The first

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author constructed for any $g \geq 2$ infinitely many tunnel number one knots whose complements contain closed incompressible surfaces of genus g [E1][E2], and for any $g \geq 1$ and $h \geq 1$, infinitely many tunnel number one knots whose complements contain an essential meridional surface of genus g and $2h$ boundary components [E2]. An interesting problem would be to determine, characterize or classify all tunnel number one knots which admit a closed or meridional incompressible surface. With this objective we initiated studying incompressible surfaces in a special class of tunnel number one knots, that of $(1, 1)$ -knots. A knot k is said to be a $(1, 1)$ -knot if there is a standard torus T in S^3 such that k is a 1-bridge knot with respect to T , that is, T and k intersect transversely in two points, which divide k into two arcs, and such that each arc is isotopic to an arc lying on T . Equivalently, consider $T \times I \subset S^3$, where $I = [0, 1]$; k is a $(1, 1)$ -knot if k lies in $T \times I$, so that $k \cap T \times \{0\} = k_0$ is an arc, $k \cap T \times \{1\} = k_1$ is an arc, and $k \cap T \times (0, 1)$ consists of two straight arcs, i.e., arcs which cross every torus $T \times \{x\}$ transversely. The family of $(1, 1)$ -knots is an interesting class of knots which contains torus knots, 2-bridge knots, and which has received much attention recently (see for example [CM],[CK], [GMM]). All satellite tunnel number one knots classified by Morimoto and Sakuma are in fact $(1, 1)$ -knots, and many of the knots constructed in [E1] are also $(1, 1)$ -knots. In the paper [E3] it is proved that if k is a $(1, 1)$ -knot whose complement contains a closed incompressible and meridionally incompressible surface, then k comes from the construction given in [E1].

The class of $(1, 1)$ -knots can be defined for any manifold M admitting a Heegaard decomposition of genus 1, i.e., it can be defined when $M = S^3$, $S^1 \times S^2$, or a lens space $L(p, q)$, the definition is the same, with T being a Heegaard torus.

Let $M = S^3$, $S^1 \times S^2$, or a lens space $L(p, q)$. In this paper we construct all $(1, 1)$ -knots in M having a meridional essential surface. First, we give a general construction which produces $(1, 1)$ -knots which admit a meridional essential surface (Section 2), and then prove that if a $(1, 1)$ -knot admit a meridional essential surface then it comes from the given construction (Section 3). In particular we show that for any given integers $g \geq 1$ and $h \geq 0$, there exist $(1, 1)$ -knots which admit a meridional essential surface of genus g and $2h$ boundary components. As we said before, $(1, 1)$ -knots in S^3 do not admit any meridional essential surface of genus 0, but when $M = L(p, q)$, we show that for any given integer $h \geq 1$, there exist $(1, 1)$ -knots which admit a meridional essential surface of genus 0 and $2h$ boundary components.

We remark that the knots constructed here are different from the ones in [E2]; in fact, most of the knots constructed in [E2] do not seem to be $(1, 1)$ -knots.

Throughout, 3-manifolds and surfaces are assumed to be compact and orientable. If X is contained in a 3-manifold M , then $\eta(X)$ denotes a regular neighborhood of X in M .

2. CONSTRUCTION OF THE MERIDIONAL SURFACE

In this section we construct meridional surfaces for $(1, 1)$ -knots. We construct the surfaces by pieces, that is, we define some surfaces in a product $T \times I$ or in a solid

torus, getting 6 types of basic pieces, denoted by $\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{\mathcal{D}}, \tilde{\mathcal{E}}, \tilde{\mathcal{F}}$. By assembling the pieces appropriately we get a $(1, 1)$ -knot having an essential meridional surface.

2.1 Let T be a torus, and for any two real numbers a, b , $a < b$, define $N_{a,b} = T \times [a, b]$, and $\partial_a N_{a,b} = T \times \{a\}$, $\partial_b N_{a,b} = T \times \{b\}$. A properly embedded arc in $N_{a,b}$ is straight if it intersects each torus $T \times \{x\}$ in one point, for all $a \leq x \leq b$. Let A be a once punctured annulus (i.e. a pair of pants) properly embedded in $N_{a,b}$, such that a boundary component of A , say $\partial_0 A$, and which we call the puncture, lies on $\partial_a N_{a,b}$ and is a trivial curve in this torus; the other components, denoted by $\partial_1 A$, are a pair of essential curves on $\partial_b N_{a,b}$. Assume that A has been isotoped so that A has only one saddle singularity with respect to the projection $N_{a,b} \rightarrow [a, b]$, that is, there is a real number y , $a < y < b$, so that $A \cap (T \times \{z\})$ consists of a trivial curve in $T \times \{z\}$ for $a \leq z < y$, $A \cap (T \times \{z\})$ consists of two essential curves in $T \times \{z\}$ for $y < z \leq b$, and $A \cap (T \times \{z\})$ consists of a curve with a selfintersection if $z = y$. Two such punctured annuli A_1, A_2 are parallel if $A_1 \times I$ is embedded in $N_{a,b}$ such that $A_1 \times \{0\} = A_1$, $A_1 \times \{1\} = A_2$, and $\partial_0 A_1 \times I$ ($\partial_1 A_1 \times I$) lies on $\partial_a N_{a,b}$ ($\partial_b N_{a,b}$).

We defined in the introduction the notion of meridional incompressibility of a surface with respect to a knot in a 3-manifold, but note that the same definition can be given with respect to a link or with respect to a collection of arcs properly embedded in a 3-manifold.

2.2 Pieces of type $\tilde{\mathcal{A}}$.

Let $N_{a,c} = N_{a,b} \cup N_{b,c}$, for $a < b < c$. Denote by A_1, \dots, A_r a collection of disjoint, parallel, once punctured annuli properly embedded in $N_{a,b}$ as in 2.1. Note that the curves $\partial_0 A_i$ are a collection of r trivial, nested curves in $\partial_a N$, i.e., each of them bounds a disk D_i , such that $D_i \subset D_{i+1}$, for $i \in \{1, 2, \dots, r-1\}$. The curves $\partial_1 A_i$ are a collection of $2r$ parallel essential curves in $\partial_b N$. Let A'_1, \dots, A'_r be another collection of properly embedded parallel once punctured annuli in $N_{b,c}$, again as in 2.1, but such that the curves $\partial_0 A'_i$, i.e. the punctures, lie on $\partial_c N_{b,c}$ and bound disks D'_i , and the curves $\partial_1 A'_i$ lie on $\partial_b N_{b,c}$. Suppose furthermore that $\{\partial_1 A_i\} = \{\partial_1 A'_i\}$. Note that the boundaries of an annulus in one side may be identified to curves belonging to two different annuli in the other side. Let \mathcal{A}' be the union of the annuli A_i and A'_i . \mathcal{A}' is a surface properly embedded in $N_{a,c}$. Note that each component of \mathcal{A}' is a torus with an even number of punctures, and that \mathcal{A}' could be connected. Let $\eta(\partial_b N_{a,b})$ be a regular neighborhood of $\partial_b N_{a,b}$ in $N_{a,c}$ such that $\eta(\partial_b N_{a,b}) \cap \mathcal{A}'$ consists of a collection of $2r$ parallel vertical annuli. Let $t_{a,c}$ be two straight arcs properly embedded in $N_{a,c}$ such that $t_{a,c} \cap \partial_a N_{a,c}$ consists of two points contained in D_1 , and similarly $t_{a,c} \cap \partial_c N_{a,c}$ consists of two points contained in D'_1 . Suppose also that the arcs of $t_{a,c}$ intersect \mathcal{A}' in finitely many points, all of which lie in $\eta(\partial_b N_{a,b})$. Assume these intersection points lie at different heights; denote the points and the height at which they occur by x_1, x_2, \dots, x_n , which are ordered according to their heights. It should be clear from the context whether we refer to an intersection point or to a level in which the point lies.

2.2.1 Let $\mathcal{A} = \mathcal{A}' - \text{int } \eta(t_{a,c})$. We call a product $N_{a,c}$ together with a surface \mathcal{A} and a pair of arcs $t_{a,c}$, a piece of type $\tilde{\mathcal{A}}$. We assume that a piece of type $\tilde{\mathcal{A}}$ satisfies the

following:

- (1) The part of A_1 up to level x_1 , D_1 and an annulus E_1 in $T \times \{x_1\}$ bound a solid torus $N_1 \subset N_{a,x_1}$. Suppose there is no a meridian disk D of N_1 , disjoint from $t_{a,c}$, and whose boundary consists of an arc on A_1 , and one arc in E_1 .
Note that if this condition is not satisfied, and so there is such a meridian disk, then the subarc t of $t_{a,c}$ which starts at D_1 and ends at x_1 , is parallel onto A_1 , that is, there is a disk F embedded in N_1 , with $\partial F = t \cup \alpha \cup \beta$, where α is an arc on D_1 and β an arc on A_1 , and $\text{int} F \cap t_{a,c} = \emptyset$. So we can slide the arc t until is at the level of the disk D_1 , and so there is a meridional compression disk for \mathcal{A} . That is, if the condition is not satisfied, then the surface \mathcal{A} will be meridionally compressible. On the other hand, it is not difficult to see that if the arc t is parallel to an arc on A_1 , then there is a meridian disk for N_1 disjoint from $t_{a,c}$.
- (2) One arc of $t_{a,c}$ start at D_1 and arrives to the point x_1 . Let t' be the other arc of $t_{a,c}$. Assume that either (i) t' is disjoint from \mathcal{A}' ; or (ii) x_1 and the first point of intersection of t' with \mathcal{A}' , say x_j , lie on different vertical annuli of $\eta(\partial_b N_{a,b}) \cap \mathcal{A}'$; or (iii) x_j and x_1 lie on the same vertical annulus, but there is no disk $D \subset \eta(\partial_b N_{a,b})$, so that $\partial D = \alpha \cup \beta \cup \gamma$, where $\alpha \subset t'$, $\beta \subset \mathcal{A}'$, which connects x_j with a point x'_1 at level $T \times \{x_1\}$, γ is an arc on $T \times \{x_1\}$, and $\text{int} D$ is disjoint from both $t_{a,c}$ and \mathcal{A}' . In other words, t' cannot be slid to lie at level $T \times \{x_1\}$.
In particular, if the first point of intersection of t' with \mathcal{A}' is x_2 , then we are assuming that x_1 and x_2 lie on different vertical annuli. Note that if this condition is not satisfied then condition (1) would be meaningless, i.e., we could always slide the arcs and find a meridional compression disk for \mathcal{A} .
- (3) Suppose a subarc of $t_{a,c}$ intersects a vertical annulus in two points, say x_i, x_j , but does not intersect any other annulus between these two points. Then this subarc, say β , is not parallel to an arc on the vertical annulus. That is, if D is any disk in $N_{a,c}$ with interior disjoint from \mathcal{A}' , such that $\partial D = \alpha \cup \beta$, where α is an arc on the vertical annulus, then $t_{a,c}$ necessarily intersects $\text{int} D$. If this condition is not satisfied then the surface \mathcal{A} will be clearly compressible.
- (4) For the part of A'_1 between levels $T \times \{x_n\}$ and $T \times \{c\}$ we have a similar condition as in (1).
- (5) For D'_1 and the arcs adjacent to it we have a similar condition as in (2).
- (6) In case $t_{a,c}$ is disjoint from \mathcal{A}' , we assume the following: In this case the annulus A_i is glued to the annulus A'_i . Let k be the knot obtained by joining the endpoints of $t_{a,c}$ contained in D_1 with an arc lying on D_1 , and by joining the endpoints contained on D'_1 with an arc lying on D'_1 . Then we get a knot contained in the solid torus bounded by $D_1 \cup A_1 \cup A'_1 \cup D'_1$. Assume this knot has wrapping number ≥ 2 in such a solid torus. This is required to avoid meridional compression disks.

Note that it is not difficult to construct pieces of type $\tilde{\mathcal{A}}$ where all of these conditions are satisfied. See Figure 1.

Remark. Condition 2.2.1(1) has an alternative description. Embed the solid torus N_1 in S^3 so that it is a standard solid torus and $A_1 \cap (T \times \{x_1\})$ is a preferred longitude of such solid torus. Connect the endpoints of $t_{a,c} \cap N_1$ contained in D_1 with an arc lying in D_1 . The other endpoints of $t_{a,c} \cap N_1$ lie on E_1 ; let α be the boundary of a meridian disk of N_1 , passing through these points and which is disjoint from D_1 . The endpoints of $t_{a,c} \cap N_1$ separate α into two arcs; joint these points by the arc of α that is not contained in E_1 . This defines a knot k in S^3 , which by construction has a presentation with two maxima. Note that if condition 2.2.1(1) is not satisfied, then k will be the trivial knot. So, a sufficient condition for the condition 2.2.1(1) to be satisfied is that k is a non-trivial 2-bridge knot. With a little work, it can be shown that this condition is also necessary. Observe that any 2-bridge knot can result from this construction, as shown in Figure 3.

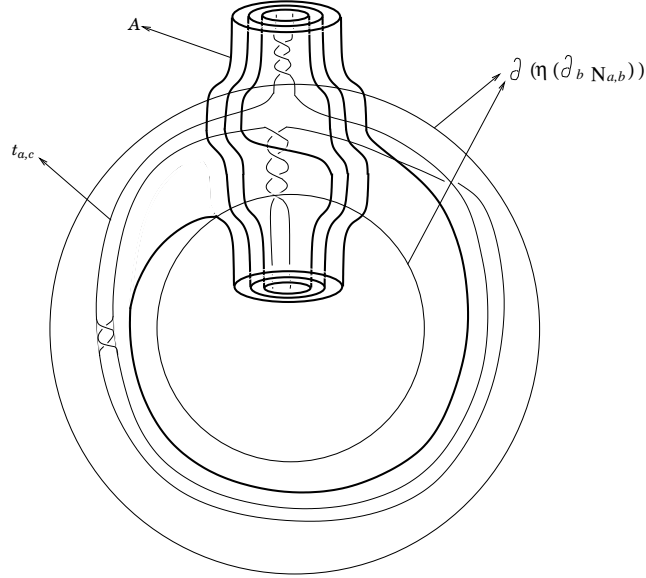


FIGURE 1

Suppose that the components of $\partial(\eta(\partial_b N_{a,b}))$, which are two tori, lie at levels x_0 and x_{n+1} , so that $a < x_0 < x_1 < \cdots < x_n < x_{n+1} < c$. Consider the torus $T \times x_0$. The annuli A_i intersect this torus in $2r$ curves which divide it into $2r$ horizontal annuli. Consider the collection of these annuli, except the one which intersects $t_{a,c}$, and denote the collection by B . Take also an horizontal annulus at a level between x_1 and x_2 , which does intersect $t_{a,c}$ in one point, and denote it by B' . But if different components of $t_{a,c}$ define the points x_1 and x_2 , i.e., a component of $t_{a,c}$ goes from D_1 to x_1 , and the other goes from D_1 to x_2 , then choose B' as an annulus between x_2 and x_3 ; in this case B' is disjoint from $t_{a,c}$. Note that in this case the points x_1 and x_2 lie on different vertical annuli, for otherwise it contradicts condition 2.2.1(2). Analogously define horizontal annuli C at level x_{n+1} and an annulus C' at a level between x_n and x_{n-1} (or between x_{n-1} and x_{n-2}). If $t_{a,c}$ is disjoint from \mathcal{A}' , then consider only the

collection of annuli B .

Lemma 2.2.2. *The surface \mathcal{A} is incompressible and meridionally incompressible in $N_{a,c} - \text{int } \eta(t_{a,c})$.*

Proof. Suppose first that the arcs $t_{a,c}$ do intersect the surface \mathcal{A}' . Suppose \mathcal{A} is compressible or meridionally compressible, and suppose that D is a compression disk for \mathcal{A} , which is disjoint from $t_{a,c}$, or intersects it in one point. Assume D intersects transversely the annuli B, B', C, C' . Let γ be a simple closed curve of intersection between D and the annuli which is innermost in D , and then it bounds a disk D' . Note that γ must be trivial in the corresponding annulus, for the core of the annulus is an essential curve in $N_{a,c}$, and then γ bounds a disk D'' in such annulus. If D'' intersects the arcs $t_{a,c}$ in two points, then $D' \cup D''$ bounds a 3-ball containing part of the arcs, and one subarc of $t_{a,c}$ will have endpoints on D'' , which is not possible. So the disk D'' is disjoint from the arcs $t_{a,c}$ or intersects them in one point, depending if the innermost disk in D intersects or not the arcs $t_{a,c}$; but in any case, by doing an isotopy this intersection curve is eliminated.

Suppose then that the intersection consists only of arcs, and let γ be an outermost arc in D , which we may assume cuts off a disk D' from D which is disjoint from $t_{a,c}$. Let $\partial D' = \gamma \cup \alpha$. The arc γ lies in one of the annuli, and suppose first it is not an spanning arc in the corresponding annulus. So γ bounds a disk E in such annulus. If E is contained in one of the annuli B or C , then E and $t_{a,c}$ are disjoint, and then by cutting D with E (or with an innermost disk lying on E), we get another compression disk having fewer intersections with the annuli. Suppose then that E lies in B' or in C' . If E is disjoint from $t_{a,c}$, the same argument applies, so assume it intersects $t_{a,c}$. There are two cases, either D' lies in the region between B' and D_1 (or C' and D'_1), or it lies in a region bounded by B' , two vertical annuli and C' or one of the annuli C (or bounded by C' , two vertical annuli and B' or one of the annuli B).

Suppose first that D' lies in the region between B' and D_1 . Let \hat{A}_1 be the part of A_1 lying between $T \times \{a\}$ and the level determined by B' . The arc α lies on \hat{A}_1 , with endpoints on the same boundary component of \hat{A}_1 . Then α is a separating arc in \hat{A}_1 , and cuts off a subsurface E' , which is a disk or an annulus (depending if E' contains or not $\partial_0 A_1$). If E' is a disk, then $D' \cup E \cup E'$ is a sphere intersecting the arcs $t_{a,c}$ in one point, which is impossible (note that in this case the point x_1 cannot lie in E'). If E' is an annulus, then $D' \cup E \cup E' \cup D_1$ bound a 3-ball P , so that ∂P intersects the arcs $t_{a,c}$ in at least 3 points, so it must intersect them in 4 points, and then the point x_1 must lie on E' . As the arcs lie in the 3-ball P , it is clear that there is a meridian disk of the solid torus bounded by $D_1 \cup \hat{A}_1 \cup B'$ which is disjoint from the arcs $t_{a,c}$, contradicting condition 2.2.1(1). The case when D' lies in the region between C' and D'_1 can be handled in a similar way.

Suppose now that D' lies in a region bounded by B' , two vertical annuli, and C' or one of the annuli C . The arc α cuts off a disk E'' from one of the vertical annuli. The disks E , D' and E'' form a sphere which intersects $t_{a,c}$ in at least two points. If intersects it in more than 2 points, then there is a subarc of $t_{a,c}$ arc going from E to

E'' , and at least one sub arc with both endpoints on E'' . As the arcs are monotonic, these cannot be tangled, so any arc with both endpoints on E'' must be parallel to E'' , which contradicts 2.2.1(3). If the sphere $E \cup D' \cup E''$ intersects $t_{a,c}$ in two points, then there is an arc of $t_{a,c}$ which intersects E'' at the point x_j , say. We can isotope the arc so that intersects the vertical annuli at a level just below B' ; this changes the order of the intersection points x_i , so that the point x_j now becomes x_2 . Note that x_1 and x_2 lie on different vertical annuli because of condition 2.2.1(2). Now B' is disjoint from $t_{a,c}$, and by an isotopy we get a disk having fewer intersections with B' .

We have shown that the outermost arc γ in D cannot bound a disk in one of the annuli. Suppose then that γ is a spanning arc of the corresponding annulus. Remember that $\partial D' = \gamma \cup \alpha$. If γ lies in B or C , then α must be a spanning arc of A_r or A'_r , and then $\gamma \cup \alpha$ is an essential curve in $N_{a,b}$, so it cannot bound a disk. If γ lies in B' , then $\gamma \cup \alpha$ must be a meridian of the solid torus $D_1 \cup \hat{A}_1 \cup B'$, but then condition 2.2.1(1) is not satisfied. Analogously, if γ lies on C' , condition 2.1.1(4) is not satisfied.

Therefore, if there is a compression or meridional compression disk D for \mathcal{A} , it must be disjoint from the annuli B, B', C, C' . If D lies in a region bounded by two vertical annuli and B (or B') and C (or C'), then as the arcs $t_{a,c}$ are straight, it is not difficult to see that there is a subarc parallel to one of the vertical annuli, which contradicts condition 2.2.1(3). If D lies in the solid torus determined by \hat{A}_1 , B' and D_1 , then either this contradicts condition 2.2.1(1), or D is not a compression or meridional compression disk. If D lies in any of the remaining regions, then it is not difficult to see that it cannot be a compression disk.

Suppose now that the arcs $t_{a,c}$ are disjoint from \mathcal{A}' . If D is a compression disk for \mathcal{A}' which is disjoint from the arcs $t_{a,c}$ or intersects them in one point, a similar argument as above shows that D is disjoint from the annuli B , and then D must lie in the solid torus bounded by $D_1 \cup A_1 \cup A'_1 \cup D'_1$. This implies that the wrapping number of the knot k defined in 2.2.1(6) is 0 or 1, contradicting that condition. \square

2.3 Pieces of type $\tilde{\mathcal{B}}$.

Let T be a torus, and let $N_{a,b} = T \times [a, b]$. Denote by A_1, \dots, A_r a collection of disjoint, parallel, once punctured annuli properly embedded in $N_{a,b}$ as in 2.1. Note that the punctures of these annuli, $\partial_0 A_i$, are a collection of r trivial, nested curves on $\partial_a N_{a,b}$; each of them bounds a disk D_i , such that $D_i \subset D_{i+1}$, for $i \in \{1, 2, \dots, r-1\}$. The curves $\partial_1 A_i$, are a collection of $2r$ parallel essential curves on $\partial_b N_{a,b}$. Let A'_1, \dots, A'_r be a collection of properly embedded parallel annuli in a solid torus R_b , such that the boundaries of the annuli are a collection of $2r$ essential curves in ∂R_b , each one going at least twice longitudinally around R_b . Assume that A'_1 separates a solid torus $N_2 \subset R_b$, such that $N_2 \cap \partial R_b$ is an annulus parallel to A'_1 , and that the interior of N_2 does not intersect any A'_i . Identify $\partial_b N_{a,b}$ with ∂R_b , so that the collection of curves $\{\partial_1 A_i\}$ is identified with the collection $\{\partial A'_i\}$. Note that the boundaries of an annulus in one side may be identified to curves belonging to two different annuli in the other side. Let \mathcal{B}' be the union of the annuli A_i and A'_i . The surface \mathcal{B}' is properly embedded

in the solid torus $N_{a,b} \cup R_b$. Note that each component of \mathcal{B}' is a punctured torus, and that \mathcal{B}' could be connected. Let $\eta(\partial_b N_{a,b})$ be a regular neighborhood of $\partial_b N_{a,b}$ in $N_{a,b} \cup R_b$ such that $\eta(\partial_b N_{a,b}) \cap R_b$ consists of a product neighborhood $\partial R_b \times [b, b + \epsilon]$, and $\eta(\partial_b N_{a,b}) \cap \mathcal{B}'$ consists of a collection of $2r$ parallel vertical annuli. Let t be an arc properly embedded in $N_{a,b} \cup R_b$, such that: $t \cap \partial_a N_{a,b}$ consists of two points contained in D_1 , $t \cap N_{a,b}$ consists of two straight arcs properly embedded in $N_{a,b}$, $t \cap R_b$ consists of an arc contained in the product neighborhood $\partial R_b \times [b, b + \epsilon]$, which has only one minimum in there, i.e., it intersects each torus twice, except one, which intersects precisely in a tangency point. Suppose t intersects \mathcal{B}' in finitely many points, all of which lie in $\eta(\partial_b N_{a,b})$, and suppose these intersection points lie at different heights; denote the points and the height at which they occur by x_1, x_2, \dots, x_n , which are ordered according to their heights.

2.3.1 Let $\mathcal{B} = \mathcal{B}' - \text{int } \eta(t)$. We call a solid torus together with a surface \mathcal{B} and an arc t , a piece of type $\tilde{\mathcal{B}}$. We assume that a piece of type $\tilde{\mathcal{B}}$ satisfies the following:

- (1) The part of A_1 up to level x_1 satisfy the same condition as in 2.2.1(1).
- (2) The same condition as in 2.2.1(2) is satisfied.
- (3) Suppose a subarc of t , which does not contain the minimum of t , intersects a component of \mathcal{B}' in two points, say x_i, x_j , but does not intersect any other component between these two points. Then this part of the arc, say β , is not parallel to an arc α on \mathcal{B}' . That is, if D is any disk in $N_{a,b} \cup R_b$ with interior disjoint from \mathcal{B}' , such that $\partial D = \alpha \cup \beta$, $\partial D \cap \mathcal{B}' = \alpha$, then t necessarily intersects $\text{int } D$. If this condition is not satisfied then the surface \mathcal{B} will be clearly compressible.
- (4) Assume that the subarc of t which contains the minimum is contained in the solid torus N_2 , and that the point x_n lies on the curve $A'_1 \cap \partial R_b$. The endpoints of the arc $t \cap R_b$ lie on the annulus $N_2 \cap \partial R_b$. Suppose there is no a meridian disk D of N_2 , disjoint from t , and whose boundary consists of an arc on A'_1 , and one arc in ∂R_b .

Note that if this condition is not satisfied, and so there is such a meridian disk, then the subarc t' of t lying in N_2 , is parallel onto ∂N_2 , that is, there is a disk F embedded in N_2 , with $\partial F = t' \cup \alpha \cup \beta$, where α is an arc on A'_1 and β an arc on ∂R_b , and $\text{int } F \cap t' = \emptyset$. So we can slide the arc t' until it is contained in ∂R_b ; by sliding it further, we could find a compression disk for \mathcal{A} .

- (5) A condition similar to 2.2.1(4) is satisfied for x_n . That is, one subarc of t starts at the maximum of t and arrives to the point x_n . Let t' be the other subarc of t that starts at the maximum. Assume that either (i) t' is disjoint from \mathcal{B}' ; or (ii) x_n and the first point of intersection of t' with \mathcal{B}' , say x_j , lie on different vertical annuli of $\eta(\partial_b N_{a,b}) \cap \mathcal{B}'$; or (iii) x_n and x_j lie on the same vertical annulus, but there is no disk $D \subset \eta(\partial_b N_{a,b})$, with $\partial D = \alpha \cup \beta \cup \gamma$, so that $\alpha \subset t'$, $\beta \subset \mathcal{B}'$, which connects x_j with a point x'_n at level $T \times \{x_n\}$, γ is an arc on $T \times \{x_n\}$, and $\text{int } D$ is disjoint from t and from \mathcal{B}' . In other words, t' cannot be slided to lie at level $T \times \{x_n\}$.

This condition complements condition (4), for if this is not satisfied, the arcs

could be slid into a position in which (4) fails.

- (6) In case t is disjoint from \mathcal{B}' , we assume the following: In this case the annulus A_i is glued to the annulus A'_i , for all i . Let k be the knot obtained by joining the endpoints of t lying on D_1 with an arc on D_1 . Then we get a knot lying in the solid torus bounded by $A_1 \cup A'_1$. Assume that this knot has wrapping number ≥ 2 in such a solid torus. If this is not satisfied, then the surface will be compressible or meridionally compressible.

It is not difficult to construct pieces of type $\tilde{\mathcal{B}}$ where all of these conditions are satisfied. See Figure 2.

If the annulus A_1 goes just one longitudinally around the solid torus R_b , and t intersects \mathcal{B}' , then it is not difficult to see that \mathcal{B} will be compressible; if t is disjoint from \mathcal{B}' , then \mathcal{B} is incompressible but each of its components will be parallel into $\partial_a N_{a,b}$.

Remark. Condition 2.2.1(4) has an alternative description. Embed the solid torus N_2 in S^3 in a standard manner, so that $A'_1 \cap \partial R_b$ is a preferred longitude of such solid torus. Suppose the point x_n lies on the curve $A'_1 \cap \partial R_b$. As before, the endpoints of the arc $t \cap R_b$ lie on the annulus $N_2 \cap \partial R_b$ and the point x_n lies on the curve $A'_1 \cap \partial R_b$. Let α be the boundary of a meridian disk of N_2 which passes through these points. The endpoints of $t \cap R_b$ separates α into two arcs; connect them with the subarc of α which intersects A'_1 . This defines a knot k in S^3 , which by construction has a presentation with two minima. It can be shown that Condition 2.2.1(4) is satisfied if and only if k is a non-trivial 2-bridge knot. Note that any 2-bridge knot k can result from this construction, as shown in Figure 3.

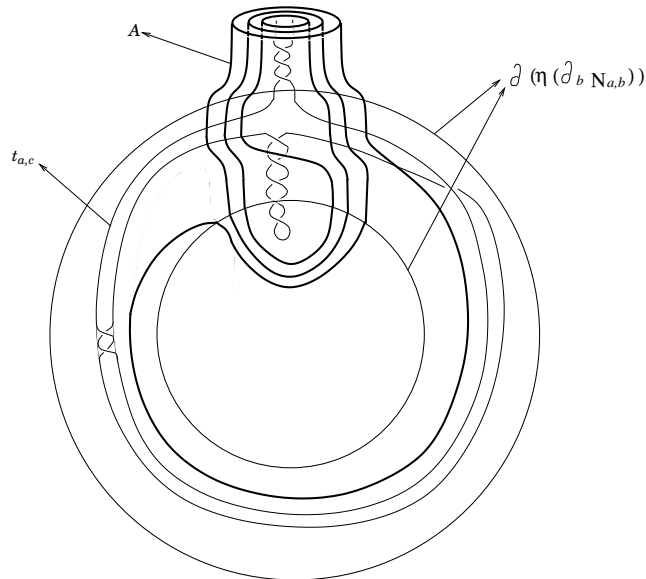


FIGURE 2

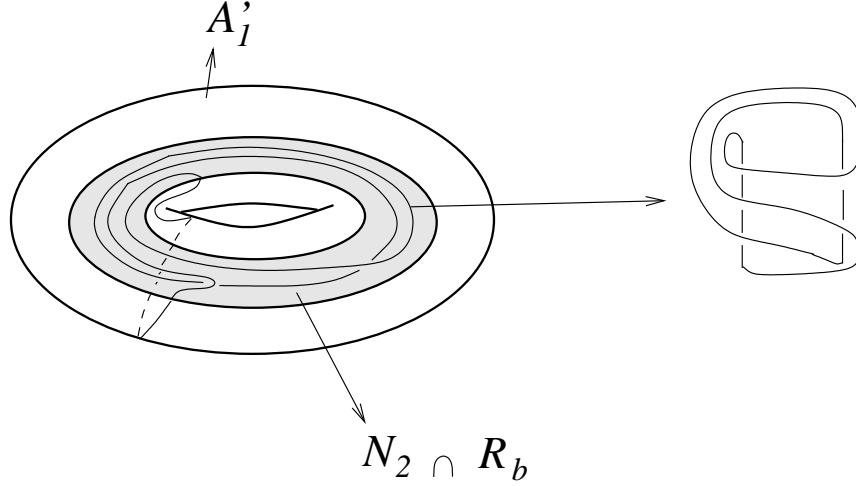


FIGURE 3

Suppose that the components of $\partial(\eta(\partial_b N_{a,b}))$ lie at levels x_0 and x_{n+1} , so that $a < x_0 < x_1 < \dots < x_n < x_{n+1} \leq b + \epsilon$. Consider the torus $T \times \{x_0\}$; note that $\mathcal{B} \cap (T \times \{x_0\})$ consist of $2r$ curves which divide that torus into $2r$ horizontal annuli. Consider the collection of these annuli, except the one which intersects t , and denote the collection by B . Take also an horizontal annulus at a level between x_1 and x_2 , which intersects t in one point, and denote it by B' . But if different subarcs of t define the points x_1 and x_2 , i.e., a subarc of t goes from D_1 to x_1 , and the other goes from D_1 to x_2 , then choose B' as an annulus between x_2 and x_3 ; in this case B' is disjoint from t , and the points x_1, x_2 lie on different vertical annuli. Analogously define horizontal annuli C at level x_{n+1} and annulus C' at a level between x_n and x_{n-1} (or between x_{n-1} and x_{n-2}). If t is disjoint from B' , then consider only the collection of annuli B .

Lemma 2.3.2. *The surface \mathcal{B} is incompressible and meridionally incompressible in $(N_{a,b} \cup R_b) - \text{int } \eta(t)$.*

Proof. Suppose first that the arc t does intersect the surface B' . Suppose D is a compression disk for \mathcal{B} , which is disjoint from t , or intersects it in one point. Suppose D intersects transversely the annuli B, B', C, C' . Let γ be a simple closed curve of intersection between D and the annuli which is innermost in D . Note that γ must be trivial in the corresponding annulus, for the core of the annulus is an essential curve in $T \times [a, b]$ or in R_b . So γ bounds a disk in such annulus, which is disjoint from the arc t or intersects it in one point, depending if the innermost disk in D intersects or not the arc t , but in any case an isotopy eliminates this intersection.

Suppose then that the intersection between D and the annuli consists only of arcs, and let γ be an outermost arc in D , which we may assume it cuts off a disk D' from D which is disjoint from t . Let $\partial D' = \gamma \cup \alpha$. The arc γ lies in one of the annuli, and suppose first it is not an spanning arc in the corresponding annulus. So γ bounds a disk E in such annulus. If E is contained in one of the annuli B or C , then E and t are disjoint, and then by cutting D with E (or with an innermost disk lying on E),

we get another compression disk having fewer intersections with the annuli. Suppose then that E lies in B' or in C' . If E is disjoint from t , the same argument as above applies, so assume it intersects t . There are three cases, either D' lies in the region between B' and D_1 ; or it lies in a region bounded by B' , two vertical annuli and C' or one of the annuli C ; or it lies in the region bounded by C' and A'_1 .

Suppose first that D' lies in the region between B' and D_1 . Let \hat{A}_1 be the part of A_1 lying between $T \times \{a\}$ and the level determined by B' . The arc α lies on \hat{A}_1 , with endpoints on the same boundary component of \hat{A}_1 . Then α is a separating arc in \hat{A}_1 , and cuts off a subsurface E' , which is a disk or an annulus (depending if E' contains or not $\partial_0 A_1$). If E' is a disk, then $D' \cup E \cup E'$ is a sphere intersecting the arcs t in one point, which is impossible (note that in this case the point x_1 cannot lie in E'). If E' is an annulus, then $D' \cup E \cup E' \cup D_1$ bound a 3-ball P , so that ∂P intersects the arcs t in at least 3 points, so it must intersect them in 4 points, and then the point x_1 must lie on E' . As the arcs lie in the 3-ball P , it is clear that there is a meridian disk of the solid torus bounded by $D_1 \cup \hat{A}_1 \cup B'$ which is disjoint from the arcs t , contradicting condition 2.3.1(1)

Suppose now that D' lies in a region bounded by B' , two vertical annuli, and C' or one of the annuli C . The arc α cuts off a disk E'' from one of the vertical annuli. The disks E , D' and E'' form a sphere which intersects t in at least two points. If intersects it in more than 2 points, then there is one subarc of t going from E to E'' , and at least one arc with both endpoints on E'' . As the arcs are monotonic, these cannot be tangled, so any arc with both endpoints on E'' must be parallel to E'' , which contradicts 2.3.1(2). If the sphere $E \cup D' \cup E''$ intersects t in two points, then there is an arc of t which intersects E'' at the point x_j , say. We can isotope the arc so that intersects the vertical annuli at a level just below B' ; this changes the order of the intersection points x_i , so that the point x_j now becomes x_2 . Note that x_1 and x_2 lie on different vertical annuli because of condition 2.3.1(2). Now B' is disjoint from t , and by an isotopy we get a disk having fewer intersections with B' .

Suppose then that D' lies in the region bounded by C' and A'_1 . The arc α cuts off a disk E' from A'_1 . Note that D' does not intersect the arc t , so E and E' both intersect t in one point, and $D' \cup E \cup E'$ bounds a 3-ball containing a subarc of t . Then there is a meridian disk of the solid torus bounded by C' and A'_1 disjoint from t , but this contradicts 2.3.1(4).

We have shown that the outermost arc γ in D cannot bound a disk in one of the annuli. Suppose then that γ is a spanning arc of the corresponding annulus. Remember that $\partial D' = \gamma \cup \alpha$. If γ lies in B or C , then α must be a spanning arc of A_r or A'_r , and then $\gamma \cup \alpha$ is an essential curve in $N_{a,b}$, so it cannot bound a disk (the curve $\gamma \cup \alpha$ could bound a disk only if A'_r were a longitudinal annulus in R_b). If γ lies in B' , then $\gamma \cup \alpha$ must be a meridian of the solid torus bounded by $D_1 \cup \hat{A}_1 \cup B'$, but then condition 2.3.1(1) is not satisfied. Analogously, if γ lies on C' , condition 2.3.1(4) is not satisfied.

Therefore, if there is a compression or meridional compression disk for \mathcal{B} , it must be disjoint from the annuli B, B', C, C' . If D lies in a region bounded by two vertical annuli and B (or B') and C (or C'), then as the arcs t are straight, it is not difficult

to see that there is a subarc parallel to one of the vertical annuli, which contradicts condition 2.3.1(3). If D lies in the solid torus determined by \hat{A}_1 , B' and D_1 , then either this contradicts condition 2.3.1(1), or D is not a compression or meridional compression disk. If D lies in any of the remaining regions, then it is not difficult to see that it cannot be a compression disk.

The remaining case is when t is disjoint from B' . In this case the annulus B' is not defined. Let D a compression or meridional compression disk. Again, we can make D disjoint from the annuli B . Then D lies in the solid torus determined by $A_1 \cup A'_1$. If ∂D is essential in that torus, this implies that the wrapping number of the knot k defined in 2.3.1(6) is 0 or 1, contradicting that condition. \square

2.4 Pieces of type \tilde{C} .

Let T be a torus, and let $N_{a,b} = T \times [a, b]$. Denote by A_1, \dots, A_r a collection of disjoint, parallel, once punctured annuli properly embedded in $N_{a,b}$ as in 2.1. Note that the punctures of these annuli, $\partial_0 A_i$, are a collection of r trivial, nested curves in $\partial_a N$, each bounding a disk D_i , and $\partial_1 A_i$ consists of a collection of $2r$ essential parallel curves on $\partial_b N_{a,b}$. Let A'_1, \dots, A'_{2r} be a collection of meridian disks in a solid torus R_b . Identify $\partial_b N_{a,b}$ with ∂R_b , so that the collection of curves $\{\partial_1 A_i\}$ are identified with the collection $\{\partial A'_i\}$, and say, the annulus A_i is glued with the disks A'_i and A'_{2r-i+1} . Let C' be the union of the annuli A_i and the disks A'_i . C' is a surface properly embedded in the solid torus $N_{a,b} \cup R_b$. Note that each component of C' is a disk. Let $\eta(\partial_b N_{a,b})$ be a regular neighborhood of $\partial_b N_{a,b}$ in $N_{a,b} \cup R_b$ such that $\eta(\partial_b N_{a,b}) \cap R_b$ consists of a product neighborhood $\partial R_b \times [b, b + \epsilon]$, and $\eta(\partial_b N_{a,b}) \cap C'$ consists of a collection of $2r$ parallel vertical annuli. Let t be an arc properly embedded in $N_{a,b} \cup R_b$, such that: $t \cap \partial_a N_{a,b}$ consists of two points contained in the disk D_1 , $t \cap N_{a,b}$ consists of two straight arcs properly embedded in $N_{a,b}$, $t \cap R_b$ consists of an arc contained in the product neighborhood $\partial R_b \times [b, b + \epsilon]$, which has only one minimum in there, i.e., it intersects each torus twice, except one, which intersects precisely in a tangency point. Assume that the arc t intersects C' in finitely many points, all of which lie in $\eta(\partial_b N_{a,b})$. Suppose these intersection points lie at different heights, are denoted by x_1, x_2, \dots, x_n , and are ordered according to their heights.

2.4.1 Let $\mathcal{C} = C' - \text{int } \eta(t)$. We call a solid torus together with a surface \mathcal{C} and an arc t , a piece of type \tilde{C} . We assume that a piece of type \tilde{C} satisfies the following:

- (1) The part of A_1 up to level x_1 satisfies the same condition as in 2.2.1(1).
- (2) A similar condition as in 2.2.1(2) is satisfied.
- (3) Suppose a subarc of t , say β , which does not contain the minimum of t , intersects a component of C' , in two points, say x_i, x_j , but does not intersects any other component between these two points. The same condition as in 2.3.1(3) is satisfied, that is, if D is any disk in $N_{a,b} \cup R_b$ with interior disjoint from C' , such that $\partial D = \alpha \cup \beta$, $\partial D \cap C' = \alpha$, then t necessarily intersects $\text{int } D$.
- (4) The subarc of t which contains the minimum has endpoints on two different disks A'_i and A'_j .

Note that in this case the arc t necessarily intersects the surface \mathcal{C}' . It is not difficult to construct examples of pieces satisfying these conditions. See Figure 4.

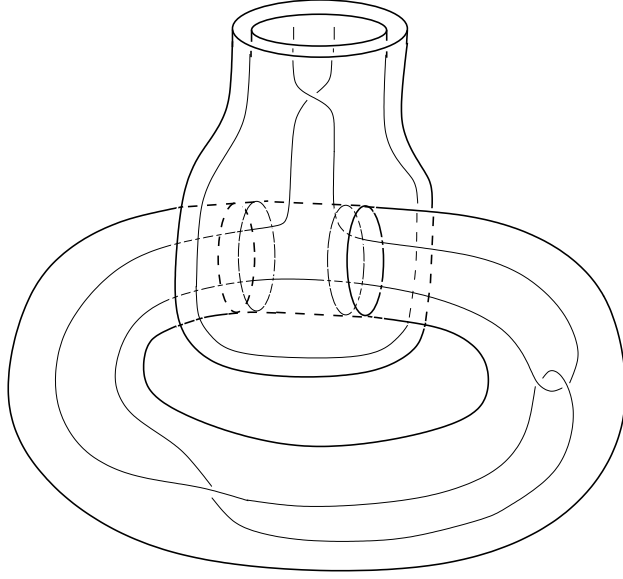


FIGURE 4

Suppose that the components of $\partial(\eta(\partial_b N_{a,b}))$ lie at levels x_0 and x_{n+1} , so that $a < x_0 < x_1 < \dots x_n < x_{n+1} \leq b + \epsilon$. Consider the torus $T \times \{x_0\}$, and note that $\mathcal{C} \cap (T \times \{x_0\})$ consists of $2r$ curves which divide that torus into $2r$ horizontal annuli. Consider the collection of these annuli, except the one which intersects t , and denote the collection by B . Note that in this case the core of any of the annuli B is a trivial curve in $N_{a,b} \cup R_b$. Take also an horizontal annulus at a level between x_1 and x_2 , which intersects t in one point, and denote it by B' . But if different subarcs of t define the points x_1 and x_2 , i.e., a subarc of t goes from D_1 to x_1 , and the other goes from D_1 to x_2 , then choose B' as an annulus between x_2 and x_3 ; in this case B' is disjoint from t , and the points x_1, x_2 lie on different vertical annuli.

Lemma 2.4.2. *The surface \mathcal{C} is incompressible and meridionally incompressible in $N_{a,b} \cup R_b - \text{int } \eta(t)$.*

Proof. Suppose D is a compression disk for \mathcal{C} , which is disjoint from t , or intersects it in one point.

Suppose first that the disk is not in the 3-ball bounded by $A_1 \cup A'_1 \cup A'_{2r}$. Suppose D intersects transversely the annuli B . Let γ be a simple closed curve of intersection between D and the annuli which is innermost in D . If γ bounds a disk in one of the annuli, then this disk is disjoint from the arc t and an isotopy eliminates this intersection. Suppose then that γ is an essential curve in one of the annuli B ; if this happens then the arc t is disjoint from a meridian of the solid torus $N_{a,b} \cup R_b$ or it intersects a meridian in one point. This implies that condition 2.4.1(3) is not satisfied,

except if t intersects each of the disks $A_i \cup A'_i \cup A'_{2r-i+1}$ in two points, but in this case it is not difficult to see that ∂D would not be essential in \mathcal{C}' .

Suppose then that the intersection between the disk D and the annuli B consists only of arcs, and let γ be an outermost arc in D , which we may assume it cuts off a disk D' from D which is disjoint from the arc t . Let $\partial D' = \gamma \cup \alpha$. The arc γ lies in one of the annuli B , and suppose first it is not an spanning arc in the corresponding annulus. So γ bounds a disk E in such annulus. As E is contained in an annulus in B , E is disjoint from t , and then by cutting D with E (or with an innermost disk lying on E), we get another compression disk having fewer intersections with the annuli. Suppose then that γ is an spanning arc of the corresponding annulus. Then α must be an arc contained in one of the components of \mathcal{C} , and the only possibility is that it is in the component formed by $A_r \cup A'_r \cup A'_{r+1}$, for it is the only component which intersects a single annulus of B in two curves. But then $\gamma \cup \alpha$ is an essential curve in $N_{a,b}$, which cannot bound a disk. If the intersection between the disk D and the annuli B is empty, then it is not difficult to see either condition 2.4.1(3) is not satisfied, or ∂D is not essential in \mathcal{C}' .

Suppose now that the disk D is contained in the ball bounded by $A_1 \cup A'_1 \cup A'_{2r}$. Assume D intersects transversely the annulus B' . Note that the annulus B' plus two disks contained in $A_1 \cup A'_1 \cup A'_{2r}$ bound a 3-ball B'' which contains all the points of intersection between t and $A_1 \cup A'_1 \cup A'_{2r}$, except x_1 (and x_2 , in case B' is disjoint from t). Suppose there is an outermost arc of intersection γ in D , which cuts off a disk D' from D , with interior disjoint from B' and t . Suppose that γ is not an spanning arc in B' ; so it cuts off a disk E from B' . If E is disjoint from t , by cutting D with E , we get a compression disk having fewer intersections with B' . So suppose that E intersects t in one point.

There are two cases, either D' lies in the region between B' and D_1 , or it lies in the 3-ball B'' . Suppose first that D' lies in the region between B' and D_1 . Let \hat{A}_1 be the part of A_1 lying between $T \times \{a\}$ and the level determined by B' . The arc α lies on \hat{A}_1 , with endpoints on the same boundary component of \hat{A}_1 . Then α is a separating arc in \hat{A}_1 , and cuts off a subsurface E' , which is a disk or an annulus (depending if E' contains or not $\partial_0 A_1$). If E' is a disk, then $D' \cup E \cup E'$ is a sphere intersecting the arcs t in one point, which is impossible (note that in this case the point x_1 cannot lie in E'). If E' is an annulus, then $D' \cup E \cup E' \cup D_1$ bound a 3-ball P , so that ∂P intersects the arcs t in at least 3 points, so it must intersect them in 4 points, and then the point x_1 must lie on E' . As the arcs lie in the 3-ball P , it is clear that there is a meridian disk of the solid tours bounded by $D_1 \cup \hat{A} \cup B'$ which is disjoint from the arcs t , contradicting condition 2.4.1(1).

Suppose now that D' lies in the 3-ball B'' . The arc α cuts off a disk E'' from one of the vertical disks. The disks E , D' and E'' form a sphere which intersects t in at least two points. If intersects it in more than 2 points, then there is a subarc of t going from E to E'' , and at least one arc with both endpoints on E'' . As the arcs are monotonic, these cannot be tangled, so any arc with both endpoints on E'' must be parallel to E'' , which contradicts 2.4.1(3). If the sphere $E \cup D' \cup E''$ intersects t in two points, then

there is a subarc of t which intersects E'' at the point x_j , say. We can isotope the arc so that intersects the vertical disk at a level just below B' ; this changes the order of the intersection points x_i , so that the point x_j now becomes x_2 . The points x_1 and x_2 lie on different vertical annuli because of condition 2.4.1(2). Now B' is disjoint from t , and by an isotopy we get a disk having fewer intersections with B' .

We have shown that the outermost arc γ in D cannot bound a disk in B' . Suppose then that γ is an spanning arc of B' . The only possibility is that D' lies in the region between B' and D_1 . Then D' must be a meridian of the solid torus bounded by $D_1 \cup \hat{A}_1 \cup B'$, but then condition 2.4.1(1) is not satisfied.

We have shown that there are no outermost arcs of intersection in D , so the intersection between D and B' is either empty or contains simple closed curves (and possibly some arcs). Let γ be a simple closed curve of intersection between D and B' which is innermost in D , and bounds a disk $D' \subset D$. If γ bounds a disk in B' , then this disk is disjoint from the arc t or intersects it in one point, depending if D' intersects or not the arc t , but in any case an isotopy eliminates this intersection. If γ is essential in the annulus B' , then D' must be contained in the 3-ball B'' , and then condition 2.4.1(3) is not satisfied, unless the arc t intersects $A_1 \cup A'_1 \cup A'_{2r}$ in 2 or 4 points. If there are two points of intersection, just x_1 and x_2 , then ∂D is either parallel to $\partial(A_1 \cup A'_1 \cup A'_{2r})$, which implies that the arc t is disjoint from $A_1 \cup A'_1 \cup A'_{2r}$, or ∂D encloses a disk containing just one of the points of t , which implies that ∂D is not essential. Suppose then that t intersects $A_1 \cup A'_1 \cup A'_{2r}$ in 4 points; in this case D' intersects t . Any other closed curve of intersection between D and B' must be concentric with γ in D , for otherwise there is an innermost curve bounding a disk disjoint from t . This implies that there is no arc of intersection between D and B' , for there will be an outermost one. So the intersection consists only of curves. As these are concentric, if there is more than one these can be removed by an isotopy. So, the intersection between D and B' consists only of the curve γ . The only possibility is that ∂D is in fact parallel to $\partial B'$, and then the disk is parallel to a disk lying on $A_1 \cup A'_1 \cup A'_{2r}$, a contradiction.

The only possibility left is that D is disjoint from B' . Then it is not difficult to see that D is parallel to a disk in $A_1 \cup A'_1 \cup A'_{2r}$, a contradiction. \square

2.5 Pieces of type \tilde{D} .

Let R_a and R_b be two solid tori. Let A_1, \dots, A_r be a collection of properly embedded parallel annuli in the solid torus R_a , such that the boundaries of the annuli are a collection of $2r$ curves in ∂R_a which go at least twice longitudinally around R_a . Let A'_1, \dots, A'_r be a collection of properly embedded parallel annuli in the solid torus R_b , such that the boundaries of the annuli are a collection of $2r$ curves in ∂R_b which go at least twice longitudinally around R_b . Identify ∂R_a with ∂R_b , so that the collection of curves $\{\partial A_i\}$ are identified with the collection of curves $\{\partial A'_i\}$. Note that the boundaries of an annulus in one side may be identified to curves belonging to two different annuli. Let \mathcal{D}' be the union of the annuli A_i and A'_i . \mathcal{D}' is a surface embedded in $R_a \cup R_b$. Note that each component of \mathcal{D}' is a torus and that \mathcal{D}' could be connected. Let $\eta(\partial R_a)$ be a regular neighborhood of ∂R_a in $R_a \cup R_b$ such that $\eta(\partial R_a) \cap R_a$ is

a product neighborhood $\partial R_a \times [-\epsilon, 0]$, and $\eta(\partial R_a) \cap R_b$ is a product neighborhood $\partial R_a \times [0, \epsilon]$. Assume that $\eta(\partial R_a) \cap \mathcal{D}'$ consists of a collection of $2r$ parallel vertical annuli. Let t be a knot embedded in $R_a \cup R_b$, such that: $t \cap R_b$ consists of an arc properly embedded in the product neighborhood $\partial R_a \times [0, \epsilon]$ containing only a minimum (as defined as in 2.3), $t \cap R_a$ consists of an arc properly embedded in the product neighborhood $\partial R_a \times [-\epsilon, 0]$, containing only a maximum (defined analogously to a minimum), and t intersects \mathcal{D}' in finitely many points, all contained in $\eta(\partial R_a) \cap \mathcal{D}'$. Suppose these intersection points lie at different heights (in $\partial R_a \times [-\epsilon, \epsilon]$), are denoted by x_1, x_2, \dots, x_n , and are ordered according to their heights.

2.5.1 Let $\mathcal{D} = \mathcal{D}' - \text{int } \eta(t)$. We call a solid torus together with a surface \mathcal{D} and an arc t , a piece of type $\tilde{\mathcal{D}}$. We assume that a piece of type $\tilde{\mathcal{D}}$ satisfies the following:

- (1) Note that A_1 separates a solid torus $N_1 \subset R_a$. Assume that the subarc of t which contains the maximum is contained in N_1 . A similar condition as in 2.3.1(4) is satisfied.
- (2) For A_1 and the minimum of t , a similar condition as in 2.3.1(5) is satisfied.
- (3) Suppose an arc, which does not contain the minimum or maximum of t , intersects a component of \mathcal{D}' in two points, say x_i, x_j , but does not intersect any other component between these two points. The same condition as in 2.3.1(3) is satisfied.
- (4) The annulus A'_1 separates a solid torus $N_2 \subset R_b$. Assume that the subarc of t which contains the minimum is contained in N_2 . The same condition as in 2.3.1(4) is satisfied.
- (5) For A'_1 and the maximum of t , a similar condition as in 2.3.1(5) is satisfied.
- (6) If t is disjoint from \mathcal{D}' , then A_i is glued to the annulus A'_i and t is a knot lying in the solid torus bounded by $A_i \cup A'_i$. In this case assume that the knot t has wrapping number ≥ 2 in such a solid torus.

Note that if one of the annuli A_i or A'_i is longitudinal in R_a or R_b , then the surface \mathcal{D} will be compressible.

The space $R_a \cup R_b$ can be S^3 , $S^1 \times S^2$ or a lens space $L(p, q)$. It is not difficult to construct examples of knots and surfaces satisfying these conditions for any of the possible spaces $R_a \cup R_b$. See Figure 5 for a specific example of such a piece in S^3 , where \mathcal{D}' is made of two annuli, i.e., $\mathcal{D}' = A_1 \cup A'_1$ (shown with gray lines), and say, R_b is the solid torus standardly embedded in S^3 , and R_a is the complementary torus, which contains the point at infinity. The knot t intersects \mathcal{D}' in two points, which divide it into two arcs; the black bold arc is the one which contains the maximum of t . Note that in this example the torus \mathcal{D}' is in fact isotopic to ∂R_a in S^3 , but \mathcal{D} is incompressible and meridionally incompressible in $S^3 - \text{int } \eta(t)$, while $\partial R_a - \text{int } \eta(t)$ is compressible.

Lemma 2.5.2. *The surface \mathcal{D} is incompressible and meridionally incompressible in $R_a \cup R_b - \text{int } \eta(t)$.*

Proof. It is similar to the proof of Lemma 2.3.2.

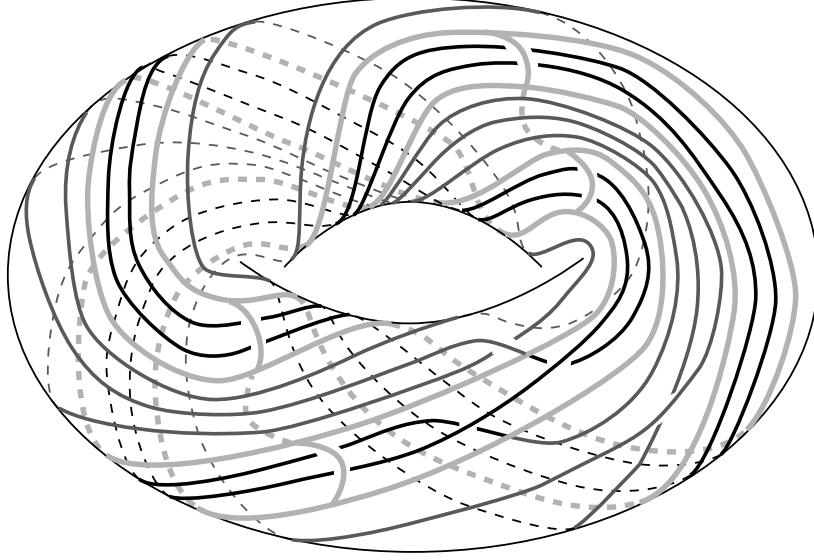


FIGURE 5

2.6 Pieces of type $\tilde{\mathcal{E}}$.

Let R_a and R_b be two solid tori. Let A_1, \dots, A_r be a collection of properly embedded parallel annuli in the solid torus R_a , such that the boundaries of the annuli are a collection of $2r$ curves in ∂R_a which go at least twice longitudinally around R_a . Let A'_1, \dots, A'_{2r} be a collection of meridian disks in a solid torus R_b . Identify ∂R_a with ∂R_b , so that the collection of curves $\{\partial A_i\}$ are identified with the collection $\{\partial A'_i\}$. Let \mathcal{E}' be the union of the annuli A_i and the disks A'_i . \mathcal{E}' is a surface embedded in $R_a \cup R_b$ and each of its components is a sphere. Note that $R_a \cup R_b$ can be any lens space $L(p, q)$, but it cannot be S^3 nor $S^1 \times S^2$. Let $\eta(\partial R_a)$ be a regular neighborhood of ∂R_a in $R_a \cup R_b$ such that $\eta(\partial R_a) \cap R_a$ is a product neighborhood $\partial R_a \times [-\epsilon, 0]$, and $\eta(\partial R_a) \cap R_b$ is a product neighborhood $\partial R_a \times [0, \epsilon]$. Assume that $\eta(\partial R_a) \cap \mathcal{E}'$ consists of a collection of $2r$ parallel vertical annuli. Let t be a knot embedded in $R_a \cup R_b$, such that: $t \cap R_b$ consists of an arc properly embedded in the product neighborhood $\partial R_a \times [0, \epsilon]$ containing only a minimum (as defined as in 2.3), $t \cap R_a$ consists of an arc properly embedded in the product neighborhood $\partial R_a \times [-\epsilon, 0]$, containing only a maximum, and t intersects \mathcal{E}' in finitely many points, all lying in $\eta(\partial R_a) \cap \mathcal{E}'$. Suppose these intersection points lie at different heights (in $\partial R_a \times [-\epsilon, \epsilon]$), are denoted by x_1, x_2, \dots, x_n , and are ordered according to its height.

2.6.1 Let $\mathcal{E} = \mathcal{E}' - \text{int } \eta(t)$. We call a solid torus together with a surface \mathcal{E} and an arc t , a piece of type $\tilde{\mathcal{E}}$. We assume that a piece of type $\tilde{\mathcal{E}}$ satisfies the following:

- (1) Note that A_1 separates a solid torus $N_1 \subset R_a$. A similar condition as in 2.3.1(4) is satisfied.
- (2) A similar condition as in 2.3.1(5) is satisfied.
- (3) Suppose an arc, which does not contain the maximum not the minimum of t ,

intersects a component of \mathcal{E}' , in two points, say x_i, x_j , but does not intersects any other component between these two points. The same condition as in 2.3.1(3) is satisfied.

- (4) The subarc of t which contains the minimum has endpoints on different disks.

It is not difficult to construct examples of pieces satisfying all these conditions. If the knot t does not intersect the surface \mathcal{E}' , then this is a sphere bounding a 3-ball in a lens space $L(p, q)$ and t is a trivial knot lying in that ball.

Lemma 2.6.2. *The surface \mathcal{E} is incompressible and meridionally incompressible in $R_a \cup R_b - \text{int } \eta(t)$.*

Proof. It is similar to the proof of Lemma 2.4.2.

Remark. We have a degenerate case of this construction, which happens when \mathcal{E} is formed by an annulus of meridional slope in R_a , and two meridians disks in R_b . In this case \mathcal{E} is a separating sphere in $S^1 \times S^2$ which bounds a 3-ball, and t is a trivial knot in such 3-ball. In this case, if t intersects \mathcal{E} , then clearly the surface will be compressible, i.e., a meridian of R_a will make a compression disk.

2.7 Pieces of type $\tilde{\mathcal{F}}$.

Let R_a and R_b be two solid tori. Let A_1, \dots, A_r be a collection of parallel meridian disk in the solid torus R_a and let A'_1, \dots, A'_r be a collection of parallel meridian disks in a solid torus R_b . Identify ∂R_a with ∂R_b , so that the collection of curves $\{\partial A_i\}$ are identified with the curves $\{\partial A'_i\}$. Let \mathcal{F}' be the union of the disks A_i and A'_i . \mathcal{F}' is a surface embedded in $R_a \cup R_b = S^1 \times S^2$. Note that each component of \mathcal{F}' is a non-separating sphere. Let $\eta(\partial R_a)$ be a regular neighborhood of ∂R_a in $R_a \cup R_b$ such that $\eta(\partial R_a) \cap R_a$ is a product neighborhood $\partial R_a \times [-\epsilon, 0]$, and $\eta(\partial R_a) \cap R_b$ is a product neighborhood $\partial R_a \times [0, \epsilon]$. Assume that $\eta(\partial R_a) \cap \mathcal{F}'$ consists of a collection of $2r$ parallel vertical annuli. Let t be a knot embedded in $R_a \cup R_b$, such that: $t \cap R_b$ consists of an arc properly embedded in the product neighborhood $\partial R_b \times [0, \epsilon]$ containing only a minimum (as defined as in 2.3), $t \cap R_a$ consists of an arc properly embedded in the product neighborhood $\partial R_a \times [-\epsilon, 0]$, containing only a maximum, and t intersects \mathcal{F}' in finitely many points, all lying in $\eta(\partial R_a) \cap \mathcal{F}'$. Suppose these intersection points lie at different heights (in $\partial R_a \times [-\epsilon, \epsilon]$), are denoted by x_1, x_2, \dots, x_n , and are ordered according to its height.

2.7.1 Let $\mathcal{F} = \mathcal{F}' - \text{int } \eta(t)$. We call a solid torus together with a surface \mathcal{F} and an arc t , a piece of type $\tilde{\mathcal{F}}$. We assume that a piece of type $\tilde{\mathcal{F}}$ satisfies the following:

- (1) Suppose an arc, which does not contain the minimum nor the maximum of t , intersects a component of \mathcal{F}' , in two points, say x_i, x_j , but does not intersects any other component between these two points. The same condition as in 2.3.1(3) is satisfied.
- (2) The subarc of t which contains the minimum (maximum) has endpoints on different disks, or in the same disk but intersect it from different sides.

It is not difficult to construct examples of pieces satisfying these conditions. If the knot t does not intersect the surface \mathcal{F}' , then t is a trivial knot in $S^1 \times S^2$, and if t intersects a component of \mathcal{F}' in one point, then t is of the form $S^1 \times \{*\}$, and any component of \mathcal{F}' is a sphere of the form $\{*\} \times S^2$.

Lemma 2.7.2. *The surface \mathcal{F} is incompressible and meridionally incompressible in $R_a \cup R_b - \text{int } \eta(t)$.*

Proof. It is similar to the proof of Lemma 2.4.2.

2.8 The meridional surface.

2.8.1 Let T be a Heegaard torus in M , and let $I = [0, 1]$. Consider $T \times I \subset M$. $T \times \{0\} = T_0$ bounds a solid torus R_0 , and $T_1 = T \times \{1\}$ bounds a solid torus R_1 , such that $M = R_0 \cup (T \times I) \cup R_1$. (If $M = S^3$, assume R_0 contains the point at infinity). Choose $2n + 1$ distinct points on I , $b_0 = 0$, $a_1, b_1, a_2, \dots, a_n, b_n = 1$, so that $a_i < b_i < a_{i+1}$, for all $i \leq n - 1$. Consider the tori $T \times \{a_i\}$ and $T \times \{b_i\}$ (so $T \times \{b_0\} = T_0$, $T \times \{b_n\} = T_1$). If β and γ are two simple closed curves on T , $\Delta(\beta, \gamma)$ denotes, as usual, its minimal geometric intersection number; if the curves lie on parallel tori, then $\Delta(\beta, \gamma)$ denotes that number once the curves have been projected into a single torus.

Denote by μ_M the essential simple closed curve on $T \times \{b_n\}$ which bounds a meridional disk in R_1 , and denote by λ_M the essential simple closed curve on $T \times \{b_0\}$ which bounds a disk in R_0 . So $\Delta(\mu_M, \lambda_M) = 1$ if $M = S^3$, $\Delta(\mu_M, \lambda_M) = 0$ if $M = S^1 \times S^2$, and $\Delta(\mu_M, \lambda_M) = p$ if $M = L(p, q)$.

Let γ_i be a simple closed essential curve embedded in the torus $T \times \{b_i\}$. Let Γ be the collection of all the curves γ_i .

Suppose that Γ satisfies the following:

- (1) Either $\gamma_0 = \lambda_M$, or $\Delta(\gamma_0, \lambda_M) \geq 2$, that is, γ_0 is not homotopic to the core of the solid torus R_0 .
- (2) $\Delta(\gamma_i, \gamma_{i+1}) \geq 2$, for all $0 \leq i \leq n - 1$.
- (3) Either $\gamma_n = \mu_M$, or $\Delta(\gamma_n, \mu_M) \geq 2$, that is, γ_n is not homotopic to the core of the solid torus R_1 .

Suppose a such $\Gamma = \{\gamma_i\}$, with $n \geq 2$ is given. If γ_0 is a meridian of R_0 , choose a piece of type $\tilde{\mathcal{C}}$ in $R_0 \cup (T \times [0, a_1])$, denoted $\tilde{\mathcal{C}}_0$, and which is determined by a surface \mathcal{C}_0 such that $\mathcal{C}_0 \cap T_0$ is a collection of curves parallel to γ_0 . If γ_0 is not a meridian of R_0 choose a piece of type $\tilde{\mathcal{B}}$ in $R_0 \cup (T \times [0, a_1])$, denoted $\tilde{\mathcal{B}}_0$, and determined by a surface \mathcal{B}_0 so that $\mathcal{B}_0 \cap T_0$ is a collection of curves parallel to γ_0 . For γ_i , $i \neq 0, n$, choose a piece of type $\tilde{\mathcal{A}}$ in $T \times [a_i, a_{i+1}]$, denoted $\tilde{\mathcal{A}}_i$, determined by a surface \mathcal{A}_i so that $\mathcal{A}_i \cap (T \times \{b_i\})$ is a collection of curves parallel to γ_i . If γ_n is a meridian of R_1 , choose a piece of type $\tilde{\mathcal{C}}$ in $R_1 \cup (T \times [a_n, 1])$, denoted $\tilde{\mathcal{C}}_n$, determined by a surface \mathcal{C}_n such that $\mathcal{C}_n \cap T_1$ is a collection of curves parallel to γ_n . If γ_n is not a meridian of R_1 choose a piece of type $\tilde{\mathcal{B}}$ in $R_1 \cup (T \times [a_n, 1])$, denoted $\tilde{\mathcal{B}}_n$, determined by a surface \mathcal{B}_n so that $\mathcal{B}_n \cap T_1$ is a collection of curves parallel to γ_n . Furthermore suppose that $\mathcal{C}_0 \cap (T \times \{a_1\}) = \mathcal{A}_1 \cap (T \times \{a_1\})$

or $\mathcal{B}_0 \cap (T \times \{a_1\}) = \mathcal{A}_1 \cap (T \times \{a_1\})$, $\mathcal{A}_i \cap (T \times \{a_{i+1}\}) = \mathcal{A}_{i+1} \cap (T \times \{a_{i+1}\})$ for $1 \leq i \leq n-1$, and $\mathcal{C}_n \cap (T \times \{a_n\}) = \mathcal{A}_n \cap (T \times \{a_n\})$ or $\mathcal{B}_n \cap (T \times \{a_n\}) = \mathcal{A}_n \cap (T \times \{a_n\})$, that is, the boundary curves of the surface in a piece are identified with the boundary curves of the surface in an adjacent piece. Suppose also that the endpoints of the arcs coincide so that the union of all the arcs is a knot K in M . Note that the union of all the surfaces becomes a surface S properly embedded in $M - \text{int } \eta(K)$, whose boundary consists of meridians of the knot K , so S is a meridional surface. It is not difficult to construct examples of surfaces satisfying those conditions.

Suppose now $\Gamma = \{\gamma_0\}$. In this case the surface is given by a piece of type $\tilde{\mathcal{D}}$, $\tilde{\mathcal{E}}$ or $\tilde{\mathcal{F}}$. More precisely, if γ_0 is not a meridian of R_0 and not a meridian of R_1 , then take a piece of type $\tilde{\mathcal{D}}$, so that the intersection of the surface \mathcal{D} with the torus $T \times \{b_0\}$ consists of curves parallel to γ_0 . Similarly, if γ_0 is a meridian of only one of R_0 or R_1 , take a piece of type $\tilde{\mathcal{E}}$, and if γ_0 is a meridian of both solid tori, then take a piece of type $\tilde{\mathcal{F}}$, or a degenerate piece of type $\tilde{\mathcal{E}}$.

Theorem 2.8.2. *Let K be a knot and S a meridional surface as constructed in 2.8.1. Then K is a $(1, 1)$ -knot and S is an essential meridional surface.*

Proof. As in 2.8.1, let T be a Heegaard torus in M , and let $I = [0, 1]$. Consider $T \times I \subset M$. $T \times \{0\} = T_0$ bounds a solid torus R_0 , and $T_1 = T \times \{1\}$ bounds a solid torus R_1 , such that $M = R_0 \cup (T \times I) \cup R_1$. As in 2.8.1, there are $2n+1$ distinct points on I , $b_0 = 0$, a_1 , b_1 , a_2, \dots, a_n , $b_n = 1$, so that $a_i < b_i < a_{i+1}$, for all $i \leq n-1$.

If S is made of one piece, i.e., it is of type $\tilde{\mathcal{D}}$, $\tilde{\mathcal{E}}$ or $\tilde{\mathcal{F}}$, then it follows from Lemmas 2.5.2, 2.6.2 and 2.7.2 that the surface is essential. Suppose then that the S is made of several pieces.

In each level a_i , consider the nested disks D_i , and between each two consecutive levels a_i, a_{i+1} , take the annuli B, B', C, C' as defined 2.2.1, 2.3.1 or 2.4.1. Suppose S is compressible in $S^3 - K$ or meridionally compressible, and let D be a compression disk which intersects K in at most one point. Assume that D intersects transversely the annuli B, B', C, C' and the disks D_i . Then the intersection consists of a finite collection of simple closed curves and arcs. Simple closed curves of intersection can be removed as in Lemmas 2.2.2, 2.3.2, 2.4.2. Suppose then that the intersections between D and the annuli and disks consists only of arcs. Let α be a such outermost arc. This can be eliminated as in those lemmas, except in two cases.

First, the arc α lies in a disk D_1 , and the disk $D' \subset D$ cutoff by α lies in a region between D_1 and an annulus B' or C' . It follows that one of the conditions 2.2.1(1), 2.3.1(1) or 2.4.1(1) is not satisfied

Second, the disk $D' \subset D$ cutoff by α lies in a region between two annuli of type B or C , i.e., a region where two different pieces of type $\tilde{\mathcal{A}}$ are glued, or where $\tilde{\mathcal{C}}_0$ or $\tilde{\mathcal{B}}_0$ are glued to \mathcal{A}_1 . The boundary of this disk will consist of an arc on lying on S and the arc α , which lies on B . Now condition 2.8.1(2) and Lemma 3.1 of [E1] show that this is not possible. Then the disk D is disjoint from the collection of annuli and disks. Again, by the lemmas, the only possibility left, is that the disk lies in a region between annuli of type B and C , and then it is disjoint from K . Again by Condition

2.8.1(2) and Lemma 3.1 of [E1] this is not possible.

Note that by construction the knot K lies in a product $T \times [-\epsilon, 1 + \epsilon]$, so that it has a maximum at $T \times \{-\epsilon\}$, a minimum at $T \times \{1 + \epsilon\}$, and intersect any other horizontal torus in 2 points. From this follows that K is a $(1, 1)$ -knot. \square

Note that if a surface S does not satisfy one of the conditions 2.8.1(1)-(3), then it will be compressible. For example, suppose that condition 2.8.1(1) is not satisfied. Then the piece $\tilde{\mathcal{B}}_0$ is made with annuli which go once longitudinally around R_0 . If t intersect \mathcal{B}_0 , it was observed in 2.3.1 that the surface will be compressible; if the knot is disjoint from \mathcal{B}_0 , then this part of the surface S is isotopic into the level torus $T \times \{a_1\}$, and then there is a compression disk lying in $T \times [a_1, b_1]$.

It follows from this construction that for any M , and any given integers $g \geq 1$ and $h \geq 0$, there exist $(1, 1)$ -knots which admit a meridional essential surface of genus g and $2h$ boundary components. In fact, there are knots which contain two or more meridional surfaces which are not parallel in the knot complement. Note that if in the construction the knot is disjoint from the surface, then what we get is one of the knots and surfaces constructed in [E1], possibly with several parallel components. As we said before, $(1, 1)$ -knots in S^3 do not admit any meridional essential surface of genus 0, but when $M = L(p, q)$, it follows that for any given integer $h \geq 1$, there exist $(1, 1)$ -knots which admit a meridional essential surface of genus 0 and $2h$ boundary components. These are given by pieces of type $\tilde{\mathcal{E}}$. In particular, there exists composite $(1, 1)$ -knots in lens spaces, i.e., knots which admit an essential meridional planar surface with two boundary components. It follows from 3.6.1(1) that these knots are obtained as a connected sum of the core of a Heegaard torus in a lens space with a two-bridge knot in the 3-sphere.

As a special case consider $(1, 1)$ -knots which admit a meridional surface of genus 1. Note that a piece of type $\tilde{\mathcal{A}}$ or of type $\tilde{\mathcal{B}}$ consists of punctured tori, so a surface S of genus 1 constructed as in 2.8.1 cannot contain two pieces of type $\tilde{\mathcal{A}}$, or two pieces of type $\tilde{\mathcal{B}}$, or a piece of type $\tilde{\mathcal{A}}$ and a piece of type $\tilde{\mathcal{B}}$. So a surface of genus 1 must consist either of a piece of type $\tilde{\mathcal{A}}$ and two pieces of type $\tilde{\mathcal{C}}$, or of a piece of type \mathcal{B} and a piece of type $\tilde{\mathcal{C}}$, or it is just a piece of type $\tilde{\mathcal{D}}$. Suppose now we have a meridional surface of genus 1 with 2 punctures in the exterior of a $(1, 1)$ -knot. Note that in a piece of type $\tilde{\mathcal{C}}$, each component always intersects the knot in at least two meridians. So if we have a meridional surface of genus 1 with two punctures, then it must consist either of a piece of type $\tilde{\mathcal{B}}$ and a piece of type $\tilde{\mathcal{C}}$, or it is a piece of type $\tilde{\mathcal{D}}$. In the first case, the piece of type $\tilde{\mathcal{B}}$ consist of just a once punctured torus without intersections with the knot, and the piece of type $\tilde{\mathcal{C}}$ consist of a disk intersecting the knot in two meridians, so the surface looks like a torus with a bubble. In the second case, the piece of type $\tilde{\mathcal{D}}$ consists of a torus which either is the boundary of a regular neighborhood of a torus knot in M , or it is isotopic (in M) to the Heegaard torus T , as in Figure 5.

It follows from [CGLS, 2.0.3] that any of the knots constructed here will have a closed incompressible surface in its complement, which will be in general meridionally compressible. However, we do not intend here to determine which is that surface.

3. CHARACTERIZATION OF MERIDIONAL SURFACES

Theorem 3.1. *Let K be a $(1, 1)$ -knot in a closed 3-manifold M , and let S be an essential meridional surface for K . Then K and S come from the construction of 2.8, that is, K and S can be isotoped so that they look as one of the knots and surfaces constructed in 2.8.*

Proof. Let T be a Heegaard torus in M , and let $I = [0, 1]$. Consider $T \times I \subset M$. $T \times \{0\} = T_0$ bounds a solid torus R_0 , and $T_1 = T \times \{1\}$ bounds a solid torus R_1 , such that $M = R_0 \cup (T \times I) \cup R_1$. Let k be a $(1, 1)$ -knot, and assume that k lies in $T \times I$, such that $k \cap T \times \{0\} = k_0$ is an arc, $k \cap T \times \{1\} = k_1$ is an arc, and $k \cap T \times (0, 1)$ consists of two straight arcs.

Suppose there is a meridional surface S in $M - \text{int } \eta(k)$, which is incompressible and meridionally incompressible. Consider S as a surface embedded in M which intersects k in a finite number of points. Assume that S intersects transversely T_0 and T_1 . Let $S_0 = S \cap R_0$, $S_1 = S \cap R_1$, and $\tilde{S} = S \cap (T \times I)$. We can assume that all the intersection points between S and k lie on \tilde{S} . Let $\pi : T \times I \rightarrow I$ be the height function, where we choose 0 to be the highest point, and 1 the lowest. We may assume that the height function on \tilde{S} is a Morse function. So there is a finite set of different points $X = \{x_1, x_2, \dots, x_m\}$ in I , so that \tilde{S} is tangent to $T \times \{x_i\}$ at exactly one point, and this singularity can be a local maximum, a local minimum, or a simple saddle. Assume also that no point of intersection between \tilde{S} and k lie on one of the levels $T \times \{x_i\}$. For any $y \notin X$, $T \times \{y\}$ intersects \tilde{S} transversely, so for any such y , $\tilde{S} \cap T \times \{y\}$ consists of a finite collection of simple closed curves called level curves, and at a saddle point x_i , either one level curve of \tilde{S} splits into two level curves, or two level curves are fused into one curve.

Define the complexity of S by the pair $c(S) = (|S_0| + |S_1| + |\tilde{S}|, |X|)$ (where $|Y|$ denotes the number of points if Y is a finite set, or the number of connected components if it is a surface, and give to such pairs the lexicographical order). Assume that S has been isotoped so that $c(S)$ is minimal.

Claim 3.2. *The surfaces S_0 , S_1 and \tilde{S} are incompressible and meridionally incompressible in R_0 , R_1 , and $T \times I - \text{int } \eta(k)$ respectively.*

Proof. Suppose one of the surfaces is compressible or meridionally compressible, say \tilde{S} , and let D be a compression disk, which it is disjoint from k , or intersects it in one point. Then ∂D is essential in \tilde{S} but inessential in S . By cutting S along D we get a surface S' and a sphere E . Note that S and S' are isotopic in $M - k$ (see the remark below the proof of this claim). For S' we can similarly define the surfaces S'_0 , S'_1 and \tilde{S}' . Note that $|S_i| = |S'_i| + |E \cap R_i|$, $i = 1, 0$, then either $|S'_0| < |S_0|$ or $|S'_1| < |S_1|$, for E intersects at least one of R_0 , R_1 . Also $|\tilde{S}'| \leq |\tilde{S}|$, so $c(S') < c(S)$, but this contradicts the choice of S . \square

Remark. The surfaces S and S' could be non-isotopic if one of the following cases occur: a) There is a non-separating sphere disjoint from the knot; or b) there is a

separating sphere intersecting the knot in two points, which bounds a 3-ball containing a non-prime knotted arc of the knot. In the first case it is not difficult to see that the knot has to be trivial, and it follows from the proof of Theorem 3.1 that the second case is impossible, for any such sphere determines a piece of type $\tilde{\mathcal{E}}$ which has a prime summand, in fact a 2-bridge knot summand.

This implies that S_0 is a collection of trivial disks, meridian disks and essential annuli in R_0 . If a component of S_0 is a trivial disk E , then ∂E bounds a disk on T_0 which contains k_0 , for otherwise $|S_0|$ could be reduced. If a component of S_0 is an essential annulus A , then A is parallel to an annulus $A' \subset T_0$, and A' must contain k_0 , for otherwise $|S_0|$ could be reduced. This implies that the slope of ∂A cannot consist of one meridian and several longitudes, for in this case A would also be parallel to $T_0 - A'$, and then $|S_0|$ could be reduced. This also implies that S_0 cannot contain both essential annuli and meridian disks. A similar thing can be said for S_1 .

Claim 3.3. *\tilde{S} does not have any local maximum or minimum.*

Proof. Suppose \tilde{S} does have a maximum. It will be shown that \tilde{S} has a component which is either a disk, an annulus, a once punctured annulus, or a once punctured torus which is parallel to a subsurface in T_1 . Choose the maximum at lowest level, say at level x_i , so that there are no other maxima between x_i and 1. So $\tilde{S} \cap (T \times \{x_i\})$ consists of a point and a collection of simple closed curves. Just below the level $\{x_i\}$, the surface \tilde{S} intersects the level tori in simple closed curves, so below the maximum, a disk E_1 is being formed; if x_i is the last singular point then the disk E_1 will be parallel to a disk on T_1 .

Look at the next singular level x_{i+1} , $x_i < x_{i+1}$. Note that the disk E_1 may intersect k . If the singular point at x_{i+1} is a local minimum, or a saddle whose level curves are disjoint from the boundary curve of E_1 , then we can interchange the singularities. If it is a saddle point involving the curve ∂E_1 and another curve, then both singular points cancel each other, by pushing down the maximum; this can also be done if one of the arcs of k intersects E_1 , for a given arc cannot intersect E_1 more than once (for otherwise a subarc of k will be isotopic to an arc on S , and then S would be compressible). If the singular point at x_{i+1} is a saddle formed by a selfintersection of the curve ∂E_1 , then the disk E_1 below the maximum transforms into an annulus E_2 . If this singularity occurs inside a 3-ball bounded by E_1 and a disk of a level torus, then it is not difficult to see that E_2 would be compressible or meridionally compressible in $T \times [y_1, y_2]$, for certain levels y_1, y_2 , and then either \tilde{S} would be compressible or meridionally compressible or could be isotoped to reduce the number of singular points. So the singularity occurs outside such 3-ball, and then the annulus E_2 must be parallel to an annulus in some $T \times \{y\}$. For any y just below x_{i+1} , ∂E_2 consists of two parallel curves c_1 and c_2 . If c_1 is a trivial curve on $T \times \{y\}$ which bounds a disk disjoint from k , then by cutting \tilde{S} with such disk, we get a surface S' isotopic to S , but with $c(S') < c(S)$, for S' has less singular points than S . If c_1 bounds a disk which intersects k once, then S would be meridionally compressible. So there remain two possibilities, either c_1 and c_2 are essential curves on $T \times \{y\}$ or c_1 bounds

a disk which intersects k twice. Note that if k intersects E_1 then necessarily c_1 and c_2 are essential curves on $T \times \{y\}$. Look at the next singular level x_{i+2} . If it is a local minimum or a saddle whose level curves are disjoint from the annulus E_2 , then again we can interchange the singularities (note that these curves cannot be inside the solid torus bounded by E_2 and an annulus in $T \times \{x_{i+2}\}$, for there would be a maximum in there). If it is a saddle joining the annulus E_2 with another curve, then by pushing E_2 down the singular point x_{i+2} is eliminated. Again note that this can also be done if one of the arcs of k intersects E_2 , for a given arc cannot intersect E_2 more than once, for otherwise S would be compressible. So this singular point has to be of the annulus with itself, and then a new surface E_3 is formed. The surface E_3 has to be a once punctured torus or a once punctured annulus. Suppose first that the surface E_3 is not parallel to some $T \times \{y'\}$, i.e., the singular point x_{i+2} occurs inside the solid torus bounded by E_2 and an annulus in $T \times \{x_{i+2}\}$. It is not difficult to see that in this case the surface E_3 is compressible or meridionally compressible; this is because the arcs of k are straight, and if they look complicated inside E_2 , we can find a level preserving isotopy which make them look “straight”. Then the surface E_3 is parallel to some surface in $T \times \{y'\}$. Note that if the knot intersects E_2 then this surface will be compressible or meridionally compressible. If not, then there may be one more singular point of E_3 with itself, but then the surface E_4 which will be formed would be compressible, for a component of its boundary would bound a disk in some $T \times \{y''\}$ disjoint from k .

We conclude that either \tilde{S} is compressible, meridionally compressible, the number of singular points is not minimal, or a component of \tilde{S} , say S' , it is parallel to a surface in T_1 . In the latter case, if the surface S' is disjoint from k , then it can be pushed to R_1 , reducing $c(S)$. If it not disjoint from k , then by pushing S' into T_1 , we see that the arc k_1 would be parallel to S , implying that S is compressible. \square

At a nonsingular level y , $\tilde{S} \cap (T \times \{y\})$ consists of simple closed curves. If such a curve γ is trivial in $T \times \{y\}$, then it bounds a disk in that torus. If such a disk is disjoint from k , then by the incompressibility of \tilde{S} , γ bounds a disk in \tilde{S} , which give rise to a single maximum or minimum, contradicting Claim 3.3. Such a disk cannot intersect k once, for S is meridionally incompressible, so γ bounds a disk which intersects k twice.

Claim 3.4. *Only the following types of saddle points are possible.*

- (1) *A saddle changing a trivial simple closed curve into two essential simple closed curves.*
- (2) *A saddle changing two parallel essential curves into a trivial curve.*

Proof. At a saddle, either one level curve of \tilde{S} splits into two level curves, or two level curves are joined into one level curve. If a level curve is trivial in the corresponding level torus and at a saddle the curve joins with itself, then the result must be two essential simple closed curves, for if the curves obtained are trivial, then \tilde{S} would be compressible or meridionally compressible. If a curve is nontrivial and at the saddle joins with itself, then the result is a curve with the same slope as the original and

a trivial curve, for the saddle must join points on the same side of the curve, by orientability. If two trivial level curves are joined into one, then because the curves must be concentric, the surface S would be compressible or meridionally compressible. So only the following types of saddle points are possible:

- (1) A saddle changing a trivial simple closed curve into two essential simple closed curves.
- (2) A saddle changing two parallel essential curves into a trivial curve.
- (3) A saddle changing an essential curve γ into a curve with the same slope as γ , and a trivial curve.
- (4) A saddle changing an essential curve γ and a trivial curve into an essential curve with the same slope as γ .

We want to show that only saddles of types 1 and 2 are possible. Look at two consecutive saddle points. We note that in many cases \tilde{S} can be isotoped so that two consecutive singular points can be put in the same level, and in that case a compression disk for \tilde{S} can be found. Namely,

- (1) A type 3 is followed by a type 1, see Figure 6(a).
- (2) A type 3 followed by a type 2, see Figure 6(b).
- (3) A type 1 is followed by a type 4, see Figure 6(c).
- (4) A type 2 followed by a type 4, see Figure 6(d).
- (5) A type 3 followed by a type 4, see Figure 6(e) and 6(f).

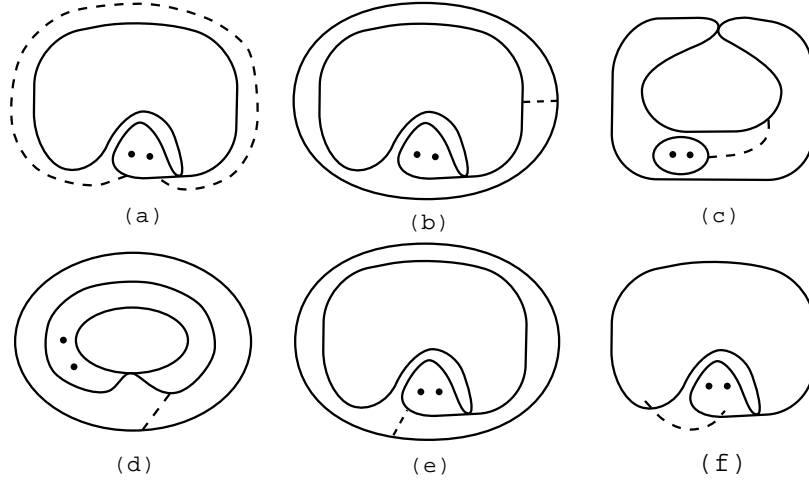


FIGURE 6

The dots in Figure 6 indicate the points of intersection between k and some $T \times \{y\}$, and the dotted line indicates the path followed by the second saddle point. Note that in all the other cases it may not be possible to put the two saddles points in the same level. Note also that the knot does not intersect \tilde{S} between two of these singular points in any of the six cases, because there is a trivial curve in each level bounding a disk which intersects k twice.

Suppose there are singularities of type 3, and take the lowest one, say at level x_m . If there is a singularity at level x_{m+1} , then it is of type 1, 2, or 4, which contradicts the previous observation. So x_m is in fact the last singularity. Again note that there is no intersections of k with the surface \tilde{S} after the singular point x_m . Then S_1 consists of at least a disk, and at least one annulus or a meridian disk. Isotope S so that the saddle at x_m is pushed into R_1 , this can be done without moving k_1 . If S_1 contained a meridian disk and a trivial disk, which joined in x_m , then now there is just one meridian disk, reducing $|S_1|$ and $|X|$ by 1, then reducing $c(S)$. If S_1 contained an annulus and a disk, which joined in the singularity, then now there is just one annulus, again reducing $c(S)$.

If there is a saddle of type 4, then take the highest one. If it is not at level x_1 , then it is preceded by a singularity of type 1, 2, or 3, which is a contradiction. So it is a level x_1 . Push the singularity to R_0 . Doing an argument as in the previous case we get a contradiction. \square

Claim 3.5. S_0 (S_1) consists only of annuli or only of meridian disks.

Proof. Suppose S_0 contains a trivial disk. Look at the first singularity. Suppose it is of type 2. If there is an annulus in S_0 , then its boundary components are joined in the saddle, but because there is the trivial disk, it is not difficult to see this implies that S is compressible. If there is a pair of meridian disks in S_0 , and their boundaries are joined in the saddle, then we get a disk which can be pushed to S_0 , reducing $|S_0|$ by 1. So the first singularity has to be of type 1, and a trivial disk touches with itself. Push the singularity to R_0 , so instead of the disk we have now an annulus, this leaves $|S_0|$ unchanged, but reduces $|X|$ by one. \square

Claim 3.6. If $|X| = 0$, then S is a piece of type \tilde{D} , \tilde{E} or \tilde{F} .

Proof. As \tilde{S} has no singular points, then it has to be a collection of annuli. Then S_0 and S_1 determine the same slope in T_0 and T_1 respectively. If S_0 and S_1 consist of annuli then we have a piece of type \tilde{D} , if one consists of annuli and the other of meridian disks then we have a piece of type \tilde{E} , and if both consist of meridian disks then we have a piece of type \tilde{F} . Note that the conditions 2.5.1, 2.6.1 or 2.7.1 for the arcs must be satisfied for otherwise the surface S would be compressible or meridionally compressible. \square

Claim 3.7. Suppose that $|X| > 0$. Then S is a meridional surface as in 2.8, and it is made of a union of pieces of type \tilde{A} , \tilde{B} and \tilde{C} .

Proof. The surface S_0 consists only of annuli or only of meridian disks, so ∂S_0 consists of p essential curves. This implies that the first saddle is of type 2, changing two essential curves into a trivial curve. The knot k may intersect the surface before the first singularity, but it cannot intersect it right after that singularity, for a trivial curve is formed. If $p > 2$, then the next singularity is again of type 2, for if it is type 1, the new essential curves that are formed will have the same slope as the original curves, and then it is not difficult to see that there is a compression disk for \tilde{S} . So there will

be singularities of type 2, until no essential curve is left, so there are in total $p/2$ of these singularities. In particular this shows that p is even; this was clear if S_0 consists of annuli, but it is not clear if it consists of meridian disks. This shows that S_0 and a part of \tilde{S} form a piece of type \tilde{B} or type \tilde{C} , depending if S_0 consists of annuli or meridian disks. After the $p/2$ singularities, the intersection of S_0 with a level torus consists of $p/2$ trivial curves, which are nested, and bound a disk which intersect k in two points. The next singularity has to be of type 1, changing a trivial curve into 2 essential curves. If the next singularity after that one is of type 2, again we have a compression disk, so the singularities are all of type 1, until there is no more trivial curves. Then there are $p/2$ of such singularities. The knot may intersect the surface only after the $p/2$ singularities of type 1, i.e., after there are no trivial curves in a level, for otherwise S would be compressible or meridionally compressible. If there are no more singularities, then we get at level 1, and then S_1 consists of annuli or meridian disks, and then we have another piece of type \tilde{B} or \tilde{C} . If there are more singularities, the next $p/2$ singularities are of type 2, and then the next ones are of type 1. Note again that the knot cannot intersect the surface in the levels in which there are trivial curves of intersection. This will form a piece of type \tilde{A} . Continuing in this way, it follows that S is made of pieces of type \tilde{A} , \tilde{B} or \tilde{C} . The conditions on the curves and on the arcs must be satisfied, for otherwise the surface will be compressible or meridionally compressible. \square

This completes the proof of Theorem 3.1. \square

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